Resourses

V. Reiner "Signed Posets"
R. Stanley "A Symmetric Generalization of the Chromatic Polynomial of a Graph"
T. Zaslavsky "Signed Graph Coloring"
R. Adin et al. "Character Formulas and Descents for the Hyperoctahedrial Group"

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- An example to give intuition about Stanley 3.1
- Proof of 3.3
- Posets
- Root systems
- Signed posets
- descent sets and linear extensions

What we have done so far ...

Stanley's theorem

for a graph G with n verticies

$(-1)^n X_G(-1) = \#$ of acyclic orientations of G

generalization for signed graphs

Generalization with signed graph and B-symetric chromatic polynomial

generalization for symetric chromatic polynomial

Zaslavski's Theorem

for a signed graph Σ with n verticies

$(-1)^n X_\Sigma(-1) = \#$ of acyclic orientations of G

Stanley's Theorem (2)

$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$$

X_G symetric chromatic

e_λ elementary symetric functions

$$a_j(G) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = j}} c_\lambda$$

generalization for B-symetric chromatic polynomial

generalization for signed graphs

What we are trying to find

* $a_j(G) = \#$ of acyclic orientations with j sinks

* $\ell(\lambda) = \#$ of sets created by partition

Review

Stanley's chromatic symmetric function. [St1]

$$\boxed{X_G(x_1, x_2, \dots) := \sum_{\substack{\varkappa : V(G) \to \mathbb{N} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}}$$

Example.

$$\begin{aligned} X_{\bullet\!-\!\bullet} = \; & \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \dots \\ & x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \dots \\ & x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \dots \\ & \;\vdots \qquad\; \vdots \qquad\qquad \ddots \\ = \; & p_1^2 - p_2, \text{ where} \end{aligned}$$

Elementary Symmetric Functions

$$e_k := \sum_{1 \le j_1 < j_2 < \cdots < j_k} x_{j_1} \cdots x_{j^k}$$

Back to our example:

$$\begin{aligned} X_{\bullet\!-\!\bullet} &= \widehat{x_1x_1} + x_1x_2 + x_1x_3 + \cdots \\ &\quad + x_2x_1 + \widehat{x_2x_2} + x_2x_3 + \cdots \\ &\quad + x_3x_1 + x_3x_2 + \widehat{x_3x_3} + \cdots \\ &\qquad\qquad\qquad \vdots \\ &= 2(x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots + x_{n-1}x_n + \cdots \\ &= 2e_2 \end{aligned}$$

Note: $X_{K_k} = k! e_k$

B-symmetric chromatic function

$$Y_G(\cdots, x_{-2}, x_{-1}, x_1, x_2, \ldots) := \sum_{\substack{\kappa : V(G) \to \mathbb{Z} \setminus \{0\} \\ proper}} \prod_{v \in V(G)} x_{\kappa(v)}$$

Signed power functions: $p_{a,b} := \sum_{i \in \mathbb{Z} \setminus \{0\}} x_i^a x_{-i}^b$. Notice that $p_{a,b} = p_{b,a}$.

Example:

$$Y_{\circ -} = \cdots + x_{-2} + x_{-1} + x_1 + x_2 + \cdots = p_{1,0}$$

1

Quasi-Symmetric function

$[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}] F(x) = [x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}] F(x)$ where $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_k$

Ex. $f(x_1, x_2, x_3, x_4) = x_1^2 x_2 x_3 + x_1^2 x_2 x_4 + x_1^2 x_3 x_4 + x_2^2 x_3 x_4$

Quasi-Symmetric basis

$$Q_{S,d}(x) = Q_S(x) = \sum_{\substack{i_1 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d}$$ where S is a subset of $[d-1] := \{1, 2, \dots d-1\}$

Ex. let $d = 4$ $\quad [d-1] = \{1, 2, 3\}$ $\quad$ let $S = \{2, 3\}$

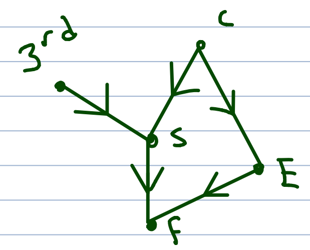
$$Q_{S,4}(x) = \sum_{\substack{i_1 \le i_2 \le i_3 \le i_4 \\ i_2 < i_3 \\ i_3 < i_4}} x_{i_1} x_{i_2} x_{i_3} x_{i_4} = \sum_{i_1 \le i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} = x_1^2 x_2 x_3 x_4 + x_1 x_2 x_5 x_{100} \cdots$$

The convention for the rest of my presentation:

$a > b$

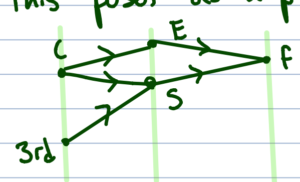
Poset - a partial order

Ex.



3rd > s
s > f
E > f
c > E
s < c

represent this poset as a picture:



For 1 graph: there is only 1 poset

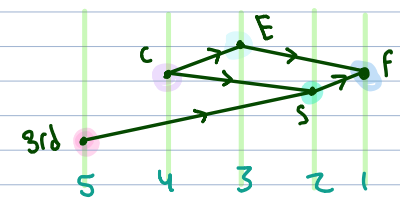
linear extension - a total order that preserves the order from the poset

3rd > C > E > S > F

linear extension as a permutation:

$$\alpha = \begin{pmatrix} 3rd & C & E & S & f \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

linear extension as a picture:

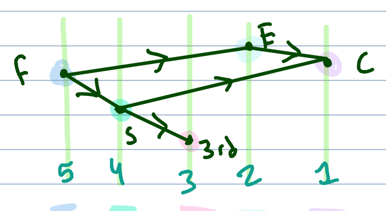


There can be 1 to many linear extensions

Order reversing linear extension

f > S > 3rd > E > C

$$\omega = \begin{pmatrix} f & S & 3rd & E & C \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$



You choose 1 ω

Descent Set (Stanley)

$$D(\alpha) = \{ j \mid a_j > a_{j+1} \}$$

$$(\omega(\alpha^{-1}(1)), \dots, \omega(\alpha^{-1}(d))) = (a_1, a_2, \dots, a_d)$$

5 4 2 1 3

$$(a_1, a_2, a_3, a_4, a_5) = (3, 5, 4, 2, 1)$$

$$= (3, 5, 4, 2, 1)$$

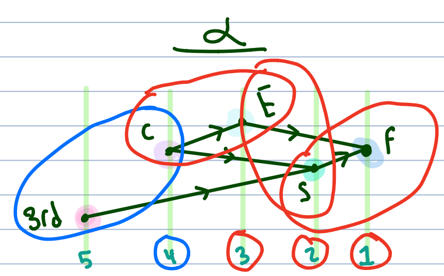
$j = 1\ \ 2\ \ 3\ \ 4\ \ 5$

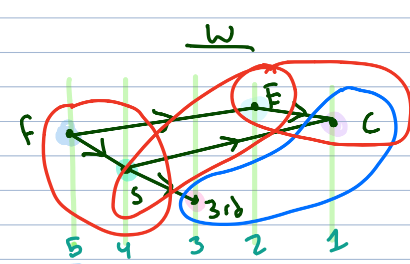
{1 2 3}

$$D(\alpha) = \{2, 3, 4\}$$

two linear extensions can have the same descent

You can think of descent sets as contradictions between α & ω





$\boxed{\mathcal{L}(P,\omega) = \{\alpha\} \text{ the set of all linear extensions}}$

Stanley defines the following:

$$X_P(x) = \sum_{\substack{\kappa \text{ compatible} \\ \text{with the orientation}}} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}$$

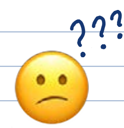
$a \longrightarrow b$

$a > b$

κ is a coloring compatible: $\kappa(a) > \kappa(b)$

Theorem 3.1

$$X_P = \sum_{\alpha \in \mathcal{L}(P,\omega)} Q_{D(\alpha)}$$



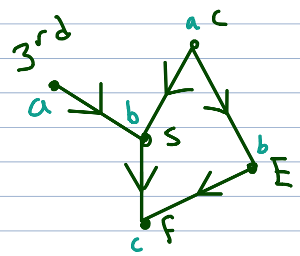
$$X_P(x) = \sum_{\substack{\kappa \text{ compatible} \\ \text{with the orientation}}} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}$$

$$X_P(x) = \sum_{\substack{\kappa(v_1) \le \kappa(v_2) \cdots \le \kappa(v_d) \\ \text{if } v_2 \longrightarrow v_1 \\ \text{then } \kappa(v_2) > \kappa(v_1)}} x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}$$

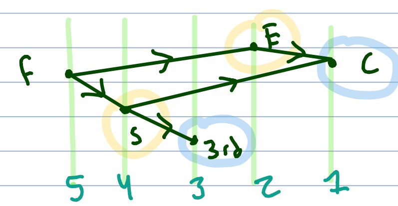
look familiar?

$$Q_S(x) = \sum_{\substack{i_1 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d}$$

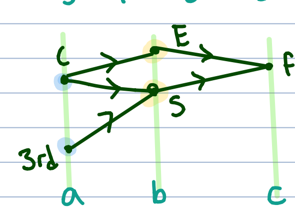
Ex.



$\omega =$



$\kappa_1 =$



Compatible coloring = POSET!

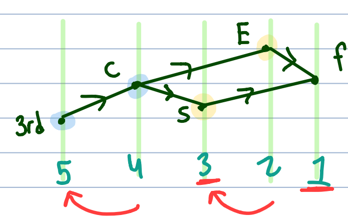
I'm going to color this graph with colors that are compatible

We have three colors: a, b, c

a > b > c

Combine κ_1 and ω to create a unique α_1

$\alpha_1 =$



$D(\alpha_1) = \{1, 3\}$

Theorem 3.3

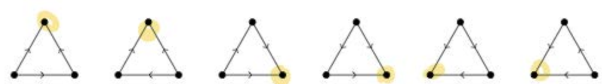
symmetric chromatic: $X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$

let $a_j(G)$ = # of acyclic orientations with j sinks

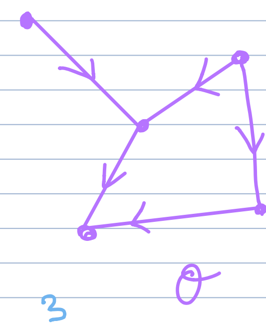
$$a_j(G) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = j}} c_\lambda$$

Ex. triangle with 1 sink has 6 acyclic orientations

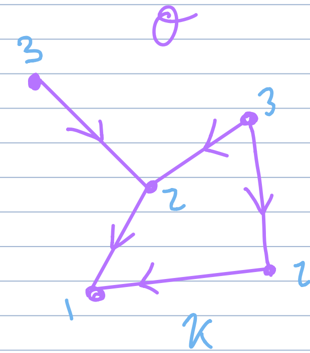
$X_\Delta = 3!\, e_3$



Proof

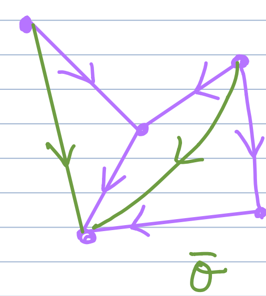


- θ is an acyclic orientation of G
- κ is a proper coloring
- θ-compatible : κ is θ-compatible if $a \longrightarrow b$, $\kappa(a) > \kappa(b)$



- Every proper coloring is compatible with one acyclic orientation
- $\mathcal{K}_\theta = \{\theta \text{ compatible proper colorings}\}$
- $\mathcal{K}_G = \{\text{all proper colorings}\}$
- $\mathcal{K}_G = \bigcup_\theta \mathcal{K}_\theta \Rightarrow X_G = \sum_\theta X_\theta$

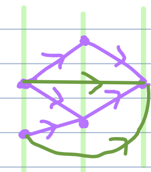
$$X_G = \sum_{\kappa \in \mathcal{K}_G} x^\kappa = \sum_\theta \sum_{\kappa \in \mathcal{K}_\theta} x^\kappa = \sum_\theta X_\theta$$



- $\bar{\theta}$ is the transitive closure of θ

notice that $\bar{\theta}$ is a poset since θ is acyclic

- $X_\theta = X_{\bar{\theta}}$ since $K_\theta = K_{\bar{\theta}}$



- Thus $X_G = \sum_\theta X_{\bar{\theta}}$

Linear transformation: $f(x+y) = f(x) + f(y)$ $f(x) = c f(x)$

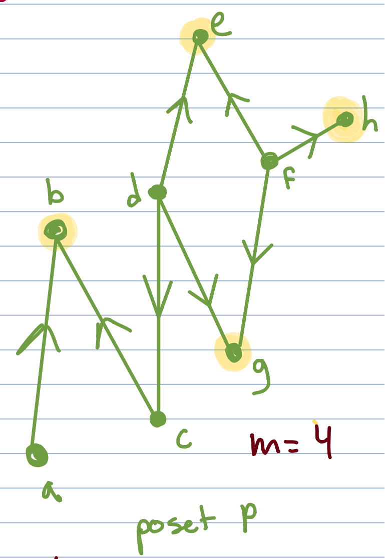
Magic

$$\varphi(Q_S) = \begin{cases} t(t-1)^i & \text{if } S = \{i+1, i+2, \dots d-1\} \\ 0 & \text{otherwise} \end{cases}$$

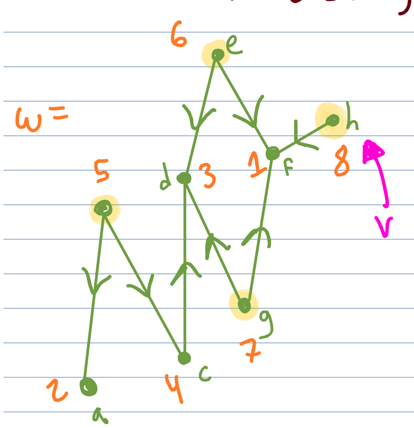
Claim: for any d element poset $\varphi(X_P) = t^m$

m = number of minimal elements

Proof of Claim:



- ★ ω is order reversing bijection
- ★ steps to get linear extension $\alpha = (a_1, \dots, a_d)$ with descent set $\{i+1, i+2, \dots d-1\}$
  1. v is the minimal element of P with largest $w(v)$ value
  2. choose any i minimal elements of P that are not v list in increasing order of lables
  3. list v
  4. list remaining elements of P in decreasing order of lables



i	possible linear extensions	descent sets
i = 3	5 6 7 8 4 3 2 1	{4, 5, 6, 7}
i = 2	6 7 8 5 4 3 2 1	{3, 4, 5, 6, 7}
i = 2	5 7 8 6 4 3 2 1	{3, 4, 5, 6, 7}
i = 2	5 6 8 7 4 3 2 1	{3, 4, 5, 6, 7}
i = 1	5 8 7 6 4 3 2 1	{2, 3, 4, 5, 6 7}
i = 1	6 8 7 5 4 3 2 1	{2, 3, 4, 5, 6 7}
i = 1	7 8 6 5 4 3 2 1	{2, 3, 4, 5, 6 7}
i = 0	8 7 6 5 4 3 2 1	{1, 2, 3, 4, 5, 6 7}
	1 2 3 4 5 6 7 8	

$\binom{4-1}{3} = 1$

$\binom{4-1}{2} = 3$

$\binom{4-1}{1} = 3$

$\binom{4-1}{0} = 1$

★ There are $\binom{m-1}{i}$ choices for $v_1, \dots, v_i$

Thus $\varphi(X_p) = \sum_{i=0}^{m-1} \binom{m-1}{i} t(t-1)^i$

Since $X_p = \sum_{\alpha \in \mathcal{L}(P,w)} Q_{D(\alpha)}$ by 3.1

and there are m unique descent sets that follow $S = \{i+1, i+2, \dots d-1\}$

and $\sum_{i=0}^{m-1} \binom{m-1}{i}$ corresponding linear extension

$$\varphi(Q_S) = \begin{cases} t(t-1)^i & \text{if } S = \{i+1, i+2, \dots d-1\} \\ 0 & \text{otherwise} \end{cases}$$

We make it !!!

$$\varphi(X_p) = \sum_{i=0}^{m-1} \binom{m-1}{i} t(t-1)^i = t^m$$

Lets finish up 3.3 :

$$X_G = \sum_{\theta} X_{\bar{\theta}}$$

if θ is acyclic then the number of minimal elements is the number of sinks

$$\varphi(X_G) = \varphi\left(\sum_{\theta} X_{\bar{\theta}}\right) = \sum_j a_j(G)\, t^j$$

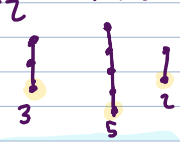
$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda$$

let P_λ be the poset which is disjoint union of chains of cardinalities $\lambda_1, \lambda_2, \dots \lambda_j$

Thus $X_{P_\lambda} = e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots$ thus $\varphi(e_\lambda) = t^{\ell(\lambda)}$

$\lambda = \{1\,2\,3 \,|\, 4\,5\,6\,7\,8 \,|\, 9\,10\}$

$\lambda_1 = 3$ $\lambda_2 = 5$ $\lambda_3 = 2$



$\ell(\lambda) = 3$

$$\varphi(X_G) = \varphi\Big(\sum_{\lambda \vdash d} c_\lambda e_\lambda\Big) = \sum_{\lambda} c_\lambda t^{\ell(\lambda)}$$

$$\sum_{\lambda \vdash d} c_\lambda t^{\ell(\lambda)} = \sum_j a_j(G)\, t^j$$

$$a_j(G) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = j}} c_\lambda$$